# Basic tariff theory

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These notes regard abstract tariff theory. The notes develop tariff theory's essential line of reasoning by standard, basic, single-variable calculus techniques.

To treat the tariff problem in its sparest form let us consider a hypothetical country that produced and consumed two goods. The prices of such goods on the global market might in general reach any levels but, to simplify the model, let us suppose that global prices were stable and that the units of trade were such that one could barter a single unit of good 1 for a single unit of good 2 at will, and vice versa. Unrealistically, to allow free-trade theory its cleanest attack at the model, let us disregard costs of transport, inventory, spoilage and the like. Further let us disregard any effect the country's global trade might have on global prices, again to allow free-trade theory its cleanest attack. Also, let us ignore monetary policy. (Are such assumptions reflective of reality? Answer: no, they are not sharply reflective, especially for the United States whose currency serves as the international reserve currency; but that is not the point of these notes. Readers who wish to do so naturally may elaborate these notes' equations with additional terms but they will find that such elaboration merely embellishes the notes' basic finding.)

Let $Q$ represent the country's total production capacity and let $x_1$ and $x_2$ represent the country's production respectively of the two goods. Many "production frontier" equations are possible. For our purpose, the equation

$$a_1 x_1^2 + a_2 x_2^2 = Q^2 \tag{1}$$

implies a production frontier that looks more or less like the curves economists usually sketch in textbooks, and is as good an equation as any for our present purpose.[1] In the simplest case, since the two goods trade at parity on global

[1]Economists call it "the quadratic additive production model."

markets, we should quantify the country's income as $x_1 + x_2 = I$. However, since the economic effects of tariffs interest us, we shall have have to elaborate the model slightly, using the equation

$$x_1 + (1+\mu)x_2 = I \tag{2}$$

instead, where $I$ represents income and $\mu$ represents the tariff rate. Here it is implicitly supposed that the country produces good 1 at comparative advantage, such that the country exports good 1 to pay for imports of good 2.

The derivative of (1) is

$$a_1 x_1 \, dx_1 + a_2 x_2 \, dx_2 = 0 \tag{3}$$

The derivative of (2) is

$$dx_1 + (1+\mu)\, dx_2 = dI \tag{4}$$

Multiplying (4) by $a_1 x_1$ or, alternately, by $a_2 x_2/(1+\mu)$ and subtracting (3) yields

$$\begin{aligned} [a_1(1+\mu)x_1 - a_2 x_2]\, dx_2 &= a_1 x_1 \, dI \\ [a_2 x_2 - a_1(1+\mu)x_1]\, dx_1 &= a_2 x_2 \, dI \end{aligned} \tag{5}$$

whereby income $I$ is maximized when

$$a_1(1+\mu)x_1 = a_2 x_2 \tag{6}$$

Multiplying (6) by $x_2$ or, alternately, by $-x_1/(1+\mu)$ and adding the result to (1), we find that

$$a_1 x_1 \left[x_1 + (1+\mu)x_2\right] = Q^2 = a_2 x_2 \left[x_2 + \frac{x_1}{1+\mu}\right] \tag{7}$$

Dividing (6) by $a_2$ or $a_1(1+\mu)$ and substituting the result into (7) to eliminate $x_2$ or $x_1$ yields

$$\left[1 + \frac{a_1(1+\mu)^2}{a_2}\right] a_1 x_1^2 = Q^2 = \left[1 + \frac{a_2}{a_1(1+\mu)^2}\right] a_2 x_2^2 \tag{8}$$

at maximal income. Equation (8) gives the country's optimal production rates $x_1$ and $x_2$ according to the model, at which per (2)

$$I = Q\sqrt{\frac{1}{a_1} + \frac{(1+\mu)^2}{a_2}} \tag{9}$$

computes the income derived from such production. This makes it possible to put (8) in terms of the $I$ of (9) as

$$\left[\frac{a_1 x_1 I}{Q}\right]^2 = Q^2 = \left[\frac{a_2 x_2 I}{Q(1+\mu)}\right]^2$$

from which

$$\begin{aligned} x_1 &= \frac{Q^2}{a_1 I} \\ x_2 &= \frac{Q^2(1+\mu)}{a_2 I} \end{aligned} \tag{10}$$

are optimal.

Goods 1 and 2 are produced for consumption, of course, or for trade followed by consumption. If $y_1$ and $y_2$ represent the hypothetical country's private consumption of goods 1 and 2, then

$$y_1 + (1+\mu)y_2 = I - T_I \tag{11}$$

where $T_I$ represents an income tax or—equivalently for this model's restricted purpose (not for all purposes)—a consumption tax the country's government collects. (Here, it is well to observe that any tariff the country might require her own government to pay when the government buys goods for official use is merely recaptured by the same government. Hence, one way or the other, unlike the country's private sector which must pay tariffs at rate $\mu$, the government effectively enjoys untariffed global market prices.) If utility—that is, how much good consumption of the two goods does the private consumer—is defined by the typical function

$$\ln U = b_1 \ln y_1 + b_2 \ln y_2 \tag{12}$$

where "ln" denotes the natural logarithmic function, then the derivative of (11) is

$$dy_1 + (1+\mu)\,dy_2 = 0 \tag{13}$$

and the derivative of (12) is

$$d(\ln U) = \frac{b_1}{y_1}\,dy_1 + \frac{b_2}{y_2}\,dy_2 \tag{14}$$

Combining (13) and (14) to eliminate $dy_2$ or $dy_1$ yields

$$\left[\frac{b_1}{y_1} - \frac{b_2}{(1+\mu)y_2}\right] dy_1 = d(\ln U) = \left[\frac{b_2}{y_2} - \frac{b_1(1+\mu)}{y_1}\right] dy_2 \tag{15}$$

whereby utility $U$ is maximized when

$$b_1(1+\mu)y_2 = b_2 y_1 \tag{16}$$

Combining this with (11) to eliminate $y_2$ or $y_1$, we find that

$$\begin{aligned} y_1 &= \frac{b_1}{b_1+b_2}(I - T_I) \\ y_2 &= \frac{b_2}{b_1+b_2}\left(\frac{I - T_I}{1+\mu}\right) \end{aligned} \tag{17}$$

at maximal utility, such that (17) gives the country's optimal consumption rates $y_1$ and $y_2$ according to the model, whereupon (12) gives the optimal utility.

On the other hand, the logarithm of (16) is

$$\ln y_2 = \ln y_1 - \ln\left[\frac{b_1}{b_2}(1+\mu)\right]$$

Applying this equation to (12) we can express optimal utility as

$$\ln U = (b_1+b_2)\ln y_1 - b_2 \ln\left[\frac{b_1}{b_2}(1+\mu)\right]$$

By the same token, applying (16) directly to (11) yields

$$\frac{b_1+b_2}{b_1} y_1 = I - T_I$$

Combining the last two equations to eliminate $y_1$, we have that

$$\ln U = (b_1+b_2)\ln\left[\frac{b_1}{b_1+b_2}(I - T_I)\right] - b_2 \ln\left[\frac{b_1}{b_2}(1+\mu)\right] \tag{18}$$

in which the optimal income $I$ is given in terms of the tariff rate $\mu$ by (9). Equation (18) thus expresses optimal utility $U$ as a function of the tariff rate $\mu$ and of the income- and consumption-tax burden $T_I$.

That tariffs, in and of themselves, substantially suppress utility is unquestioned to the extent to which the model is valid. However, the chief reason tariffs suppress utility is that tariffs are taxes. All taxes, in and of themselves, substantially suppress utility. Income and consumption taxes substantially suppress utility, too. The question thus is not whether a tariff substantially suppresses utility but rather *whether it suppresses utility substantially more than does an equivalent income or consumption tax.* For we cannot do without taxes altogether.

Let $T$ represent the total tax—income, consumption and tariff—the country's government will levy on the country's private sector to fund government operations. If the tariff level is $\mu$ then

$$T_\mu = \mu(y_2 - x_2) \tag{19}$$

(as observed in an earlier paragraph, it is implicitly assumed that $y_2 \geq x_2$ such that good 2 is imported rather than exported). The total tax then consists of

$$T = T_I + T_\mu \tag{20}$$

Given the constraint implied by (20), whereby income and consumption taxes can fall as tariffs rise (some will irritably dispute that Congress would actually allow income taxes to fall, but analytical notes like these cannot treat such hypotheticals), the question is: what happens to utility $U$ when a tariff $\mu$ is imposed?

To answer the question, we begin by combining (19) and (20) to express the tax constraint in the form

$$T = T_I + \mu(y_2 - x_2)$$

Equations (10) and (17) provide expressions respectively for $x_2$ and $y_2$. Using these in the last equation gives

$$T = T_I + \mu\left[\frac{b_2}{b_1+b_2}\left(\frac{I-T_I}{1+\mu}\right) - \frac{Q^2(1+\mu)}{a_2 I}\right]$$

which, solved for $T_I$, is

$$T_I = \frac{T(1+\mu) + \mu\left[\frac{[Q(1+\mu)]^2}{a_2 I} - \frac{b_2}{b_1+b_2} I\right]}{1 + \frac{b_1}{b_1+b_2}\mu} \tag{21}$$

In combination with (9) and (18) this answers the question, and completes the development of the basic theory.

Now, we would like to know whether and how the introduction of a tariff in a country that had not had one marginally affected the country's utility according to (18). To find out, we can compute and evaluate the derivative of (18) to be

$$\left[\frac{d(\ln U)}{d\mu}\right]_{\mu=0} = (b_1+b_2)\frac{(dI/d\mu)-(dT_I/d\mu)}{I-T_I} - b_2\bigg|_{\mu=0}$$

to which (9), (20) and (21) give

$$\begin{aligned}\left[\frac{dI}{d\mu}\right]_{\mu=0} &= \frac{Q^2}{a_2 I}\bigg|_{\mu=0} \\ \left[\frac{dT_I}{d\mu}\right]_{\mu=0} &= \frac{Q^2}{a_2 I} - \frac{b_2}{b_1+b_2}(I-T)\bigg|_{\mu=0} \\ [T_I]_{\mu=0} &= T\end{aligned} \tag{22}$$

so

$$\left[\frac{d(\ln U)}{d\mu}\right]_{\mu=0} = b_2\frac{I-T}{I-T_I} - b_2\bigg|_{\mu=0}$$

That is,

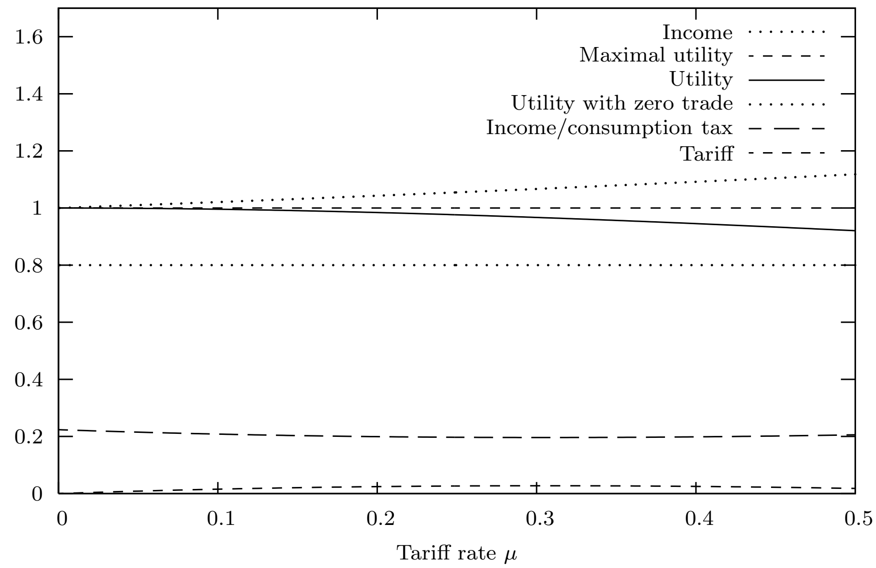
$$\left[\frac{d(\ln U)}{d\mu}\right]_{\mu=0} = 0 \tag{23}$$

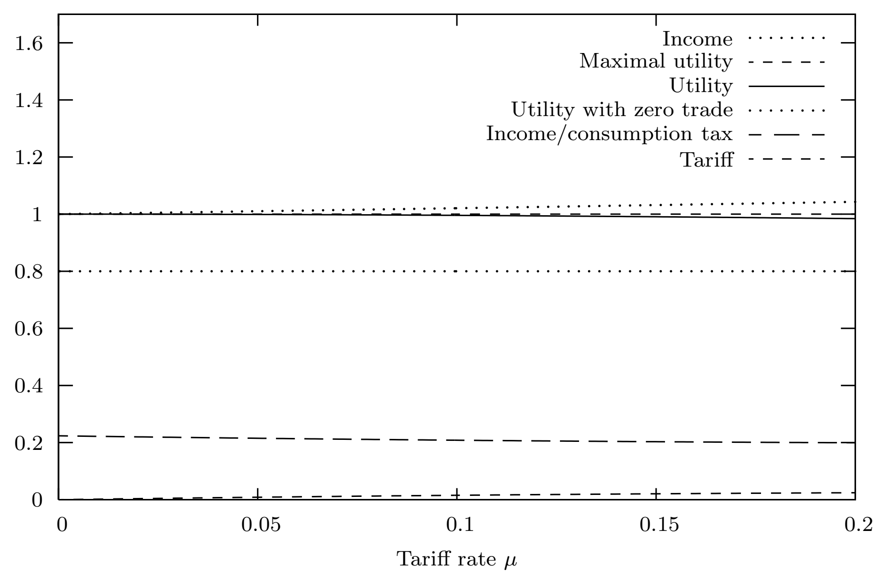
If this is correct then, *to first order,* the introduction of a tariff does not marginally affect utility at all. Any effect at $\mu = 0$ is according to the theory at most a second-order effect.

Figure 1 graphs the theoretical tariff model for one typical set of parameters. Notice how the income and tariff-revenue curves start upward right away but the utility curve is perfectly level at first, a consequence of (23). Only gradually does utility begin to slope downward.

One of the commonest errors in interpreting the theory is to fail to distinguish between tariffed trade, where $\mu > 0$, and *autarky*—which is, zero trade—where $\mu$ is irrelevant and, for these notes' purpose, $I$ is irrelevant,

Figure 1: The tariff model for $Q = 1$, $T = 1/2$, $a_1 = 1/4$, $a_2 = 1$, $b_1 = 1$, $b_2 = 1$



too.[2] A correct understanding of the theory regards the autarkical and tariffed cases altogether differently. In the autarkical case $y_1 = x_1$ and $y_2 = x_2$ if taxes are not considered and thus (12) is

$$U = x_1^{b_1} x_2^{b_2}$$

Solving (1) for $x_2^2$ and substituting the result into the square of the last equation yields

$$U^2 = x_1^{2b_1} \left( \frac{Q^2 - a_1 x_1^2}{a_2} \right)^{b_2}$$

Setting $dU/dx_1 = 0$, whereby also $d(U^2)/dx = 2U(dU/dx) = 0$,

$$0 = 2b_1 x_1^{2b_1 - 1} \left( \frac{Q^2 - a_1 x_1^2}{a_2} \right)^{b_2} - b_2 x_1^{2b_1} \left( \frac{Q^2 - a_1 x_1^2}{a_2} \right)^{b_2 - 1} \left( \frac{2a_1 x_1}{a_2} \right)$$

Dividing out common factors,

$$0 = b_1(Q^2 - a_1 x_1^2) - b_2 x_1 (a_1 x_1)$$

That is,

$$x_1 = Q \sqrt{\frac{1}{a_1} \left( \frac{b_1}{b_1 + b_2} \right)}$$

and likewise

$$x_2 = Q \sqrt{\frac{1}{a_2} \left( \frac{b_2}{b_1 + b_2} \right)}$$

Substituting these values into (12), we have that

$$\ln U_{\text{autarkical}}\big|_{T=0} = \frac{1}{2} \left[ (b_1 + b_2) \ln \frac{Q^2}{b_1 + b_2} + b_1 \ln \frac{b_1}{a_1} + b_2 \ln \frac{b_2}{a_2} \right] \tag{24}$$

[2] That $I$ should be irrelevant in the autarkical case may momentarily discomfit some readers inasmuch as $I$ represents income. However, *income* in these notes is a technical term. For income as such to play a role in these notes' model requires an internationally convertible unit of exchange.

The reader who is not persuaded by such arguments is invited to define autarkical income $I_{\text{autarkical}}$ as he sees fit and then to try to modify the theory to take it into account. The writer suspects that such a reader will find that, considering only the equations relevant to the present analysis, any such $I_{\text{autarkical}}$ must factor cleanly out of each term in every equation in which it appears—which means that one can define it but that, whether defined or not, it has zero effect on the theory's prediction of balances of production and consumption. This is why we have not bothered here to define *income* as a technical term in the autarkical case.

a different and usually significantly lower number than (18) gives without net taxes. When an income tax enters into the autarkical model it is hard for us to compare numbers directly, inasmuch as an income tax assumes an income which in the autarkical model we have, strictly speaking, not defined. Nonetheless,

$$U_{\text{autarkical}} = \frac{U|_{\mu=0}}{U|_{T=0,\mu=0}} \, U_{\text{autarkical}}|_{T=0} \tag{25}$$

seems not an unreasonable way to adjust the numbers into direct comparison. In addition to the trading figures of earlier paragraphs, Figure 1 also graphs autarkical utility so adjusted. (A subtle point regarding tariffed trade is that it is theoretically preferable to zero trade even when the tariff rate is high enough to cut off all private imports. The reason is that tariffed trade leaves the government free to import good 2 effectively at the untariffed price, whereas zero trade, autarky, has no such provision. That the government would choose directly to import precisely that which it asks the private sector not to import is probably politically absurd, of course, but this is not the point. In reality what would probably happen is that the government would acquire good 2 domestically, driving up its domestic price slightly and thus causing the private sector to import enough to supply the lack left by the government's acquisition, which importation would bring the domestic price right back down to where it had been. If we think about it, such indirection has precisely the same net theoretical effect in every respect as though the government had itself done the importing. The model as already given thus implicitly subsumes these considerations.)

In the trading case one should take care to observe the constraint that $y_2 \geq x_2$ such that good 2 is indeed imported because, if the tariff were very high, nothing *in the equations* would prevent an exporter from exporting good 2 to collect a negative tariff as a subsidy, which in reality makes no sense. In the figure, this practical limit would exist somewhere off the graph's right edge where the tariff curve crossed the horizontal axis downward.

This completes the basic development of the essentials of tariff theory, except to note that to compute the tariff rate that maximizes tariff revenue, in case one wanted to do this, makes a tedious additional exercise which the reader can pursue at his discretion, most likely by means of an appropriate application of Newton's method.

The writer would be grateful for notice of analytical errors readers might spot in these notes. Suitable references include Neil Vousden, *Economics of*

*Trade Protection,* Cambridge, 1990. (In referring, observe that the classic, unmodified Heckscher-Olin model does not fully serve these notes' purpose because the existence of an international unit of trade independent of the country in question is here assumed.)